"The" Eigenvalue-based Mixing Time Estimate

Sawyer J Robertson

October 5, 2025

This result is super well known, so I am just reproducing a proof here as a standalone reference for myself or others. See (1), for example. I think a version also appears in a book by Diaconis.

Theorem 1. *Let $(X_t)_{t\geq 0}$ be an ergodic and reversible Markov chain on a finite state space $\mathcal{X}$ with transition probability matrix P. Let*

$$1 = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \ldots \geq \lambda_n > -1$$

denote the eigenvalues of P, and let π denote the (unique) stationary state of P. For each $t > 0$ and $x_0 \in \mathcal{X}$, the total variation distance between the distribution of the process at time t after starting at x_0 and the stationary state satisfies

$$4\|P^t(x_0,\cdot) - \pi\|_{TV}^2 \leq \frac{1-\pi_{\min}}{\pi_{\min}}|\lambda_*|^{2t}$$

where $\pi_{\min} = \min_{x\in\mathcal{X}} \pi(x)$ and $\lambda_* = \max\{|\lambda_2|, |\lambda_n|\}$.

Proof of Theorem 1. Let Π denote the diagonal matrix with entries given by π. Since P is reversible, we have that the matrix $Q = \Pi^{1/2}P\Pi^{-1/2}$ is symmetric, and therefore orthogonally diagonalizable, and moreover has the same eigenvalues as P since the vector π is positive in each entry. We therefore have that

$$\begin{aligned}
4\|P^t(x_0,\cdot) - \pi\|_{TV}^2 &= \|P^t(x_0,\cdot) - \pi\|_1^2 \\
&= \|(\delta_{x_0} - \pi)P^t\|_1^2 \\
&= \|(\delta_{x_0} - \pi)\Pi^{-1/2}Q^t\Pi^{1/2}\|_1^2
\end{aligned}$$

where δ_{x_0} is the corresponding Dirac measure (expressed as a row vector) for $x_0 \in \mathcal{X}$. Now we can think of the diagonal matrix $\Pi^{1/2}$ as an operator from $(\mathbb{R}^{\mathcal{X}}, \|\cdot\|_2) \to (\mathbb{R}^{\mathcal{X}}, \|\cdot\|_1)$, and note that the norm of a diagonal operator acting between these spaces will be the ℓ_2-norm of its entries, which in this case is one. Thus we have

$$\begin{aligned}
\|(\delta_{x_0} - \pi)\Pi^{-1/2}Q^t\Pi^{1/2}\|_1^2 &\leq \|\Pi^{1/2}\|_{\ell_2\to\ell_1}^2\|(\delta_{x_0} - \pi)\Pi^{-1/2}Q^t\|_2^2 \\
&= \|(\delta_{x_0} - \pi)\Pi^{-1/2}V\Lambda^tV^T\|_2^2,
\end{aligned}$$

where we express $Q = V\Lambda V^T$ for an orthogonal matrix V and a diagonal matrix Λ consisting of the eigenvalues of Q, which coincide with those of P. Note that we may choose $v_1 = \pi^{1/2}$. Then by orthogonal invariance of the 2-norm and some linear algebra we have

$$
\begin{aligned}
\|(\delta_{x_0}-\pi)\Pi^{-1/2}V\Lambda^t V^T\|_2^2 &= \|(\delta_{x_0}-\pi)\Pi^{-1/2}V\Lambda^t\|_2^2 \\
&= |\lambda_*|^{2t}\|(\delta_{x_0}-\pi)\Pi^{-1/2}V\|_2^2 \\
&= |\lambda_*|^{2t}\sum_{j=1}^{|\mathcal{X}|}\left|\langle \pi(x_0)^{-1/2}\delta_{x_0}-\pi^{1/2}, v_j\rangle\right|^2 \\
&= |\lambda_*|^{2t}\left\{\left(\pi(x_0)^{-1/2}\pi(x_0)^{1/2}-1\right)^2+\pi(x_0)^{-1}\sum_{j=1}^{n}\left|\langle\delta_{x_0}, v_j\rangle\right|^2\right\} \\
&\le \frac{|\lambda_*|^{2t}}{\pi_{\min}}\sum_{j=1}^{n}\left|\langle\delta_{x_0}, v_j\rangle\right|^2 \\
&= \frac{|\lambda_*|^{2t}}{\pi_{\min}}(1-\pi(x_0)) \\
&\le \frac{1-\pi_{\min}}{\pi_{\min}}|\lambda_*|^{2t},
\end{aligned}
$$

from which the claim follows. □

References

[1] D. A. Levin and Y. Peres, *Markov chains and mixing times*, vol. 107, American Mathematical Soc., 2017.